Exponentials and logarithms!

As it turns out, straight lines are not the only way to model physical processes. There are many other types of equations that can be used depending on the behavior of the process being modeled. There are parabolas, cubics, or polynomials of any degree, there are the trigonometric functions (sine, cosine, etc...), and many more. The two we will focus on here are exponentials and logarithms.

I. What is an exponential function?
II. How do I tell if a process can be modeled by an exponential?
III. What is a logarithm?
IV. How can I use logarithms to find out useful things?

I. What is an exponential function?

Functions:

Before we launch into what an exponential function is, we need a working definition of a *function*. Loosely speaking, a function is a relationship between an *input variable* (usually x), and an *output variable* (usually y.) If I have a rule that assigns every input to a unique output, then I have a function.

The previous discussion on the equation of a straight line could have been phrased in terms of a linear function:

Consider the equation $y = 3x + 4$.

To write this equation as a function, we could replace y with $f(x)$ (pronounced "*f of x*"):

$$f(x) = 3x + 4.$$

Here f is the *name* of the function, and x is the *argument* of the function. If I want to know what y is for a particular value of x, I can just substitute that value for x into both sides of the equation:

$$f(2) = 3(2) + 4$$
$$f(2) = 10$$

This statement says that the line represented by this function passes through the point (2,10). The upside of this notation is that both the x and y-coordinates are written in the equation. With the previous notation, we would have to include an extra statement about the value of x:

When $x = 2$, $y = 10$.

We can now write both the *x* and *y* coordinates of various points on the line much more quickly than we could before. In fact, we are not restricted to putting in numbers; we can put in variable expressions as well!

$$f(0) = 4$$
$$f(1) = 7$$
$$f(a) = 3a + 4$$
$$f(a^2 - 7) = 3(a^2 - 7) + 4$$

In each case, we took our original expression and replaced *x* by whatever was in the parentheses.

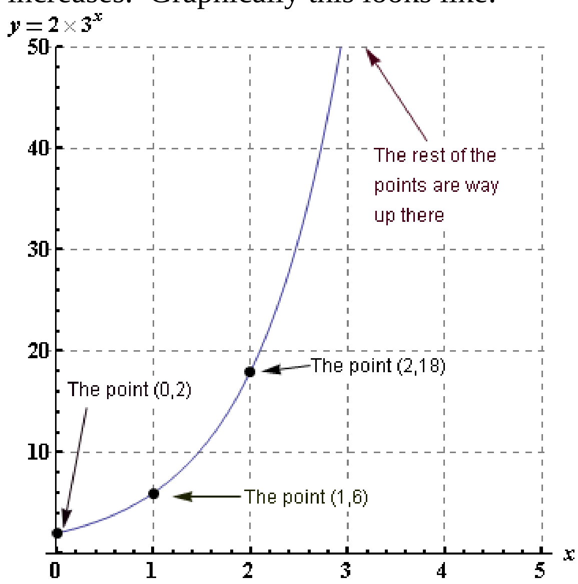
Exponential functions:
An exponential function is a function where the variable appears in the exponent. For example:

$$f(x) = 2 \cdot 3^x$$

The number being raised to the power of *x* is called the *base* of the exponential. In this case our base is 3. Let's take a look at the value of this function for various values of *x*:

$$f(0) = 2 \cdot 3^0 = 2$$
$$f(1) = 2 \cdot 3^1 = 6$$
$$f(2) = 2 \cdot 3^2 = 18$$
$$f(3) = 2 \cdot 3^3 = 54$$

We can see that the *y*-coordinates of the function begin to increase very quickly as *x* increases. Graphically this looks like:



Exponential growth and decay:
The previous example was of *exponential growth*, since the value of the function kept getting larger as the value of x increased. Any exponential function with a base that is larger than 1 will have the same basic shape. We can generalize this like so:

> An exponential function with base greater than one is called an *exponential growth* function.

What happens if the base is less than one? Let's consider the following example:

$$f(x) = 10 \cdot \left(\frac{1}{2}\right)^x$$

Let's look at some representative values of this function:

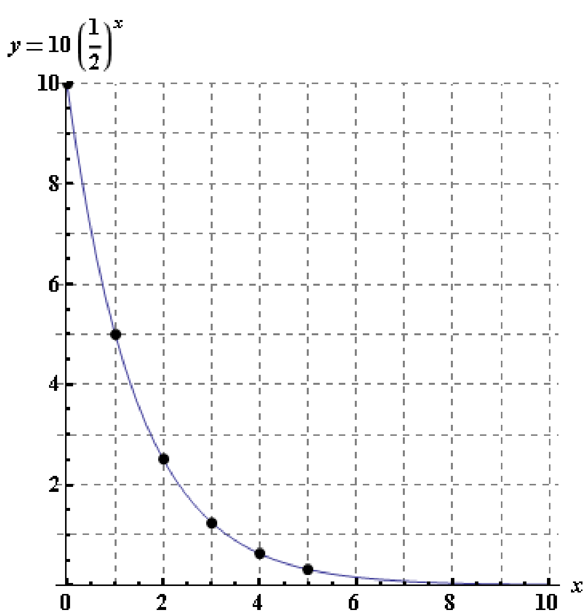
$$f(0) = 10 \cdot \left(\frac{1}{2}\right)^0 = 10$$

$$f(1) = 10 \cdot \left(\frac{1}{2}\right)^1 = 5$$

$$f(2) = 10 \cdot \left(\frac{1}{2}\right)^2 = 2.5$$

$$f(3) = 10 \cdot \left(\frac{1}{2}\right)^3 = 1.25$$

We can see that as x gets larger, the value of the function decreases. In fact, each value is half of the previous value. Here's the graph of this function:



Notice that the value of the function decreases toward zero as x gets larger. We say that this function is *decaying exponentially to zero*. The *decaying* part means that the values get smaller as x increases. The *exponentially* part means that this is an exponential function.

We can now generalize the notion of an exponential function:

> An exponential function can be written in the form: $f(x) = a \cdot b^x$.
> The function is exponential growth if $b > 1$.
> The function is exponential decay if $b < 1$.

Doubling time and half-life:
An interesting property of exponential functions is that if x is increased by a certain amount, the value of the function will double (if it is exponential growth) or get cut in half (if it is exponential decay.) The amount that x is increased in order to achieve this is called the *doubling time* (for growth) and the *half-life* (for decay.)

We can use this property to come up with a different form for the exponential function, expressed in terms of the doubling time or half-life.

> Example:
> Find a formula for a population with an initial value (that is, the value at $t = 0$) of 5 that doubles every 7 years.

If we denote the population after t years as $P(t)$, we can find the value of the function at various times:

$$P(0) = 5$$
$$P(7) = 5 \cdot 2 = 10$$
$$P(14) = 5 \cdot 2 \cdot 2 = 5 \cdot 2^2 = 20$$
$$P(21) = 5 \cdot 2 \cdot 2 \cdot 2 = 5 \cdot 2^3 = 40$$

A pattern immediately emerges. The exponent on the 2 increases by 1 every 7 years. It should not be hard to convince yourself that the exponent should be $\frac{t}{7}$. Since the 5 appears at the front of every expression, it should have a prominent place in our function as well. Just by staring at the pattern a little bit, we can see that the population function should be:

$$P(t) = 5 \cdot 2^{\frac{t}{7}}$$

If we check the value of this function for any value of t, we can see that this agrees with our model.

For a function that decays exponentially, we could go through a similar exercise. The result of this is if we are given an initial value for our function and the doubling time or half-life, we can very quickly find a formula for the function:

Exponential growth: $P(t) = P_0 \cdot 2^{\frac{t}{d}}$

Here P_0 is the initial value of the function, and *d* is the doubling time.

Exponential decay: $P(t) = P_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$

Here P_0 is the initial value of the function and *h* is the half-life.

Summary:

Exponential functions are functions where the variable is in the exponent.

They can be written in the form: $f(x) = a \cdot b^x$ where $b > 0$.

If $b > 1$, the function is exponential growth.

If $b < 1$, the function is exponential decay.

An exponential growth function can be written in terms of its *initial value* P_0 and its doubling time *d* like so:

$$P(t) = P_0 \cdot 2^{\frac{t}{d}}$$

An exponential decay function can be written in terms of its *initial value* P_0 and its half-life *h* like so:

$$P(t) = P_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{h}}$$

II. When can a process be modeled as an exponential function?

In order to determine if a set of data can be modeled as either exponential growth or decay, we need to figure out some properties of exponential functions.

Consider the function $f(x) = a \cdot b^x$:

Let's look at how this function behaves as *x* increases in increments of 1.

$$f(0) = a \cdot b^0$$
$$f(1) = a \cdot b^1$$
$$f(2) = a \cdot b^2$$
$$f(3) = a \cdot b^3$$

We can see that if *x* is increased by one, the function gets multiplied by the base:

$$f(1) = b \cdot f(0)$$
$$f(2) = b \cdot f(1)$$
$$f(3) = b \cdot f(2)$$
$$\vdots$$
$$f(n+1) = b \cdot f(n)$$

From just a minor rearrangement of the terms, we get:

$$\frac{f(1)}{f(0)} = b$$
$$\frac{f(2)}{f(1)} = b$$
$$\frac{f(3)}{f(2)} = b$$
$$\vdots$$
$$\frac{f(n+1)}{f(n)} = b$$

We can see that if we divide consecutive values of the exponential function, we always get the same number, which is just the base of the exponential function. This suggests a method for determining if a set of data can be modeled as an exponential: Divide consecutive values of the function, and see if you get the same number (or very nearly the same number) every time!

If you do get the same number, you also have the base of the exponential.

It is interesting to compare this to the way we determined if a set of data could be modeled as a straight line. In that case, we *subtracted* consecutive values of the function and checked to see if all the numbers were the same. If they were, the *common difference* between all of the terms ended up being the slope of the line.

For exponential functions, we *divided* consecutive values of the function and checked to see if all the numbers were the same. If they were, the *common ratio* between all of the terms ended up being the *base* of the exponential.

So if you have a set of data, you can determine which model is a better fit (the linear or the exponential) by comparing the difference between consecutive terms and the ratio of consecutive terms.

Summary:
To determine if a set of data can be modeled as an *exponential function*, divide consecutive values (where difference in the x values are always the same) of the function, and see if you get the same number. This number should be the value for b in the equation: $f(x) = a \cdot b^x$. Once b is known, a can by found by substituting any one of the data points into the function.

III. Logarithms
Sometimes we may wish to solve for a variable that is located in an exponent. To do so, a function was invented in order to undo exponentiation. This function is called a *logarithm*. The logarithm is defined as follows:

$$b^x = y$$
$$\log_b y = x$$

These two statements are equivalent. We can say that the logarithm is the *inverse* of the exponential function. The base of the exponential *b* becomes the *base* of the logarithm.

Here are a couple more things we can conclude from this definition:

$$\log_b(b^x) = x$$
$$b^{\log_b(x)} = x$$

There are simplification rules for logarithms analogous to those for exponentials:

$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$
$$\log_b(x^y) = y\log_b(x)$$

Let's see how this applies to something specific:

Consider the statement: $2^3 = 8$

We can rewrite this equation in terms of logs: $\log_2(8) = 3$

We can also write this in terms of the log of another base:

$$2^3 = 8$$
$$\log_{10}(2^3) = \log_{10}(8)$$
$$3\log_{10}(2) = \log_{10}(8)$$
$$3 = \frac{\log_{10}(8)}{\log_{10}(2)} = \log_2(8)$$

The bases that are used the most.

If you look at your calculator, you'll notice that there are two buttons used for finding the logarithm of a number. There is the plain *log* button (which has a base of 10) and the *ln* button (which has a base of e = 2.71828…) As a consequence, most problems are solved in terms of logs of base 10 or of base e. From this point on, if the base of the logarithm is not explicitly written with the base, we shall assume it is a log of base 10.

Here is a list of some important values for the log with a base of 10:

$$\log(1) = \log(10^0) = 0$$
$$\log(10) = \log(10^1) = 1$$
$$\log(100) = \log(10^2) = 2$$
$$\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$$
$$\log(10^x) = x$$

Let's solve a problem using this:

Solve for x in the following equation:

$$3 \cdot 7^x = 4$$

We can take the log of both sides of this equation:

$$\log(3 \cdot 7^x) = \log(4)$$

We can use the multiplication rule to turn the left hand side of the equation into the sum of two logs:

$$\log(3) + \log(7^x) = \log(4)$$

We can use the exponent rule to bring the x out of the exponent:

$$\log(3) + x\log(7) = \log(4)$$

Now we can solve for x:

$$x\log(7) = \log(4) - \log(3)$$
$$x = \frac{\log(4) - \log(3)}{\log(7)}$$

We can simplify further using the division rule:

$$x = \frac{\log\left(\frac{4}{3}\right)}{\log(7)}$$

Now we just need to know what the log of each number is, either using a calculator or looking them up in a table:

$$\log\left(\frac{4}{3}\right) = .1249$$

$$\log(7) = .8451$$

$$x = \frac{\log\left(\frac{4}{3}\right)}{\log(7)} = \frac{.1249}{.8451} = .1478$$

We can check (using a calculator) if we are right by substituting our value for *x* into the original equation:

$$3 \cdot 7^{.1478} = 3 \cdot 1.333 = 4$$

Solving for variables inside a logarithm:

We know already that a logarithm undoes an exponential. The reverse is true, too: *exponentiation undoes a logarithm.* For this reason, an exponential is sometimes referred to as an *antilog*.

It is easiest to see how this is done through an example:

Solve for *x*:

$$\log_3(2x+1) = 7$$

Since the log has a base of 3, we can undo it by raising 3 to the power of each side of the equation:

$$3^{\log_3(2x+1)} = 3^7$$

$$2x + 1 = 3^7$$

$$x = \frac{3^7 - 1}{2} = 1093$$

Notice that the logarithm allows us to solve equations we could not solve before.

Summary:

The following statements are equivalent:

$$b^x = y \text{ and } \log_b y = x$$

The logarithm and exponentials are inverses of each other:

$$\log_b(b^x) = x$$

$$b^{\log_b(x)} = x$$

The important rules for simplifying a log are:

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b(x^y) = y\log_b(x)$$

If the base of a logarithm is not explicitly written, we can assume the base is 10.
The *natural logarithm*, denoted by ln(x), has a base of e = 2.71828…

We can convert a logarithm of any base into logs of base 10 like so:

$$\log_b(x) = \frac{\log(x)}{\log(b)}$$

IV. Applications of Logarithms

Many functions encountered in nature are so-called *power laws*. These are functions that can be expressed in the form $f(x) = a \cdot x^k$, where a and k are constants that can be determined from experimental data. If we are given a set of data relating two quantities, and we are reasonably sure that the function describing this relationship is a power law, then we can use logarithms to determine the parameters a and k.

i. *How can we know if our function is a power law in the first place?*

There is no sure fire way to determine this based on data alone, but a convincing condition would be if $f(0) = 0$. There are many physical scenarios where this must be trivially true. For example, if you drop a ball from a height of *0*, then the time it will take to hit the ground will be *0*. Therefore it would be reasonable to assume that the function relating height and time is a power law.

ii. *How do we use logs to determine the parameters of a power law?*

Here we need to define something called a *log-log plot*. Instead of graphing y vs. x, we can graph $\log(y)$ vs. $\log(x)$. If the function is a power law, then the *log-log plot* will be a straight line, and the slope of the line will be the exponent k.

Here's how it works: Start with the original expression:

$$y = a \cdot x^k$$

Now, take the log of both sides (any base will do, we'll use base 10.)

$$\log(y) = \log(a) + k\log(x)$$

We can substitute new variables into this expression as follows:

$$Y = \log(y)$$
$$X = \log(x)$$

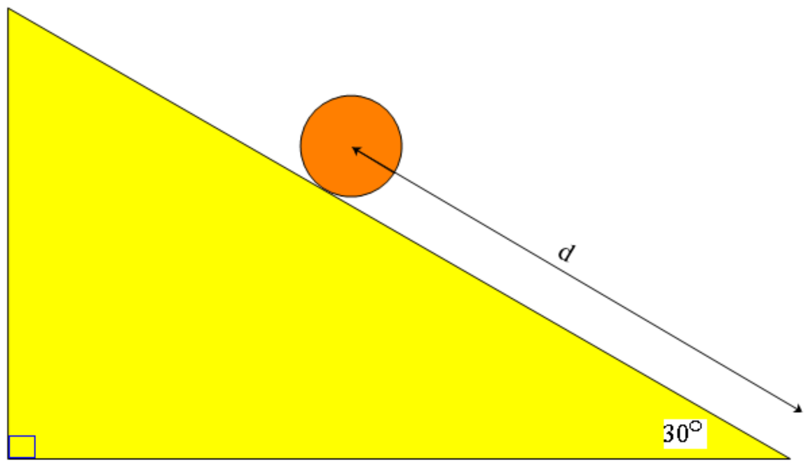
The new expression is:

$$Y = \log(a) + kX$$

This is simply the equation of a straight line with slope k and y-intercept $\log(a)$. So with a *log-log plot* of our data, we can determine the parameters of the power law by finding the slope and y-intercept of the graph.

Example: Rolling a ball bearing down a ramp.

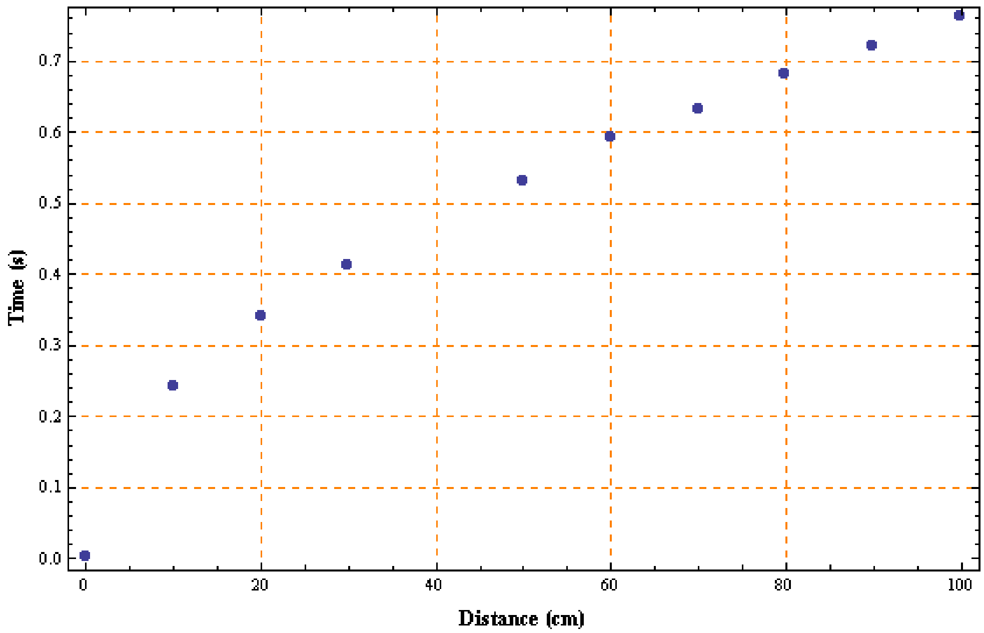
Suppose you performed the following experiment: You take a ramp that is inclined 30 degrees from the horizontal and roll a ball bearing down it (starting from rest) from various starting positions. For each starting position, you time how long it takes the ball bearing to reach the bottom. Here is a cartoon of the setup:



Suppose this is the data collected from the experiment:

Distance (cm)	Time (s)
0	0
10	.24
20	.34
30	.41
40	.48
50	.53
60	.59
70	.63
80	.68
90	.72
100	.76

The goal of the experiment is to find a function relating the distance the ball rolls (*d*) and the time it takes to get to the bottom (*t*). It could be helpful to see these points plotted on a graph:

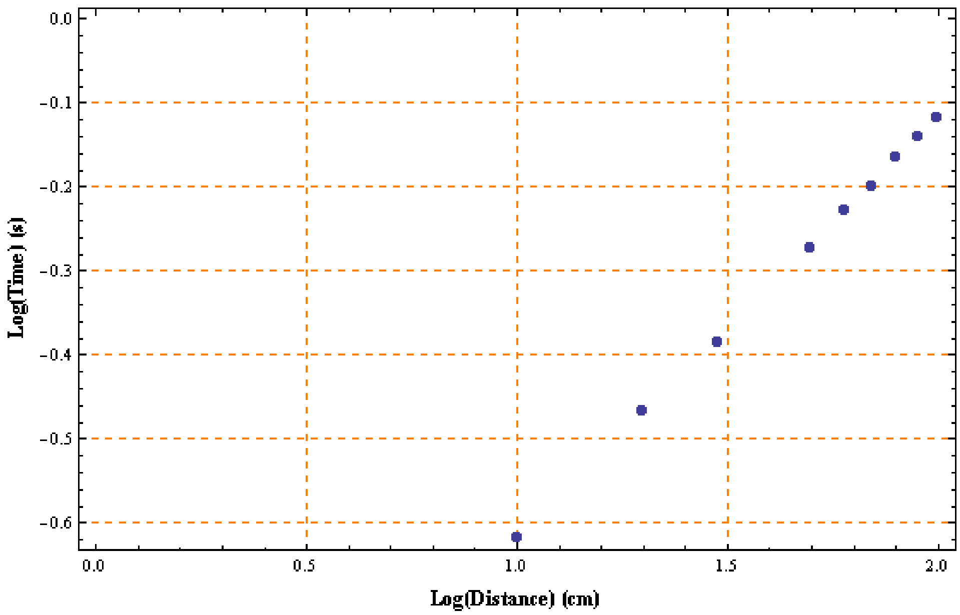


It is a little hard to tell from the graph, but this does not look like a straight line. Since the graph goes through the origin, it seems reasonable to suspect that the function governing the relationship between distance and time is a power law. We can determine this by looking at a *log-log plot* of the data:

Here is a table of the same data, except the entries will be the log of the entries in the previous table: We shall omit the (0,0) entry, since log(0) doesn't exist.

Log(Distance)	Log(Time)
1	-.62
1.3	-.47
1.48	-.39
1.6	-.32
1.7	-.28
1.78	-.23
1.85	-.20
1.9	-.17
1.95	-.14
2	-.12

If we plot graph this data, we see something that looks much more like a straight line. All that is left to do is find the equation of the line, and we'll have our power law!



A good estimate for the equation of this line will use the first and last points: (1,-.62) and (2,-.12).

You can verify that the equation of this line is: $y = .5x - 1.12$.

This means that the exponent in our power law (*k)* is .5, and the log of the constant multiplier (*a*) is -1.12. We can solve for *a* relatively quickly:

$$\log(a) = -1.12$$
$$a = 10^{-1.12}$$
$$a = .076$$

The function relating the distance the ball rolled and the time the ball took to roll down the ramp is thus:

$$t = .076 \cdot d^{0.5}$$

This function should allow us to make predictions for the time the ball will take to roll down the ramp starting from *any* point on the ramp!

Summary:

- A function is called a *power law* if it is of the form $y = a \cdot x^k$

- If a set of data relating two quantities is a power law, then a *log-log plot* of the data will form a straight line.

- The slope of the line will be *k*, and the y-intercept of the line will be log(*a*).

Extra Notes:

The data used for the example was more precise than what you would find in an actual experiment. A real experiment would have "noise" due to unforeseen factors such as friction, drag, and just plain old human measurement error. The *log-log plot* of the data in such an event would be *almost* linear in such a case, and the power law could then be approximated by using a technique called *linear regression*. This technique would find the equation of the straight line that best fits the data, and from there the power law can be inferred.