Straight Lines!

As you may or may not already know, one way to write the equation of a straight line is $y = mx + b$.

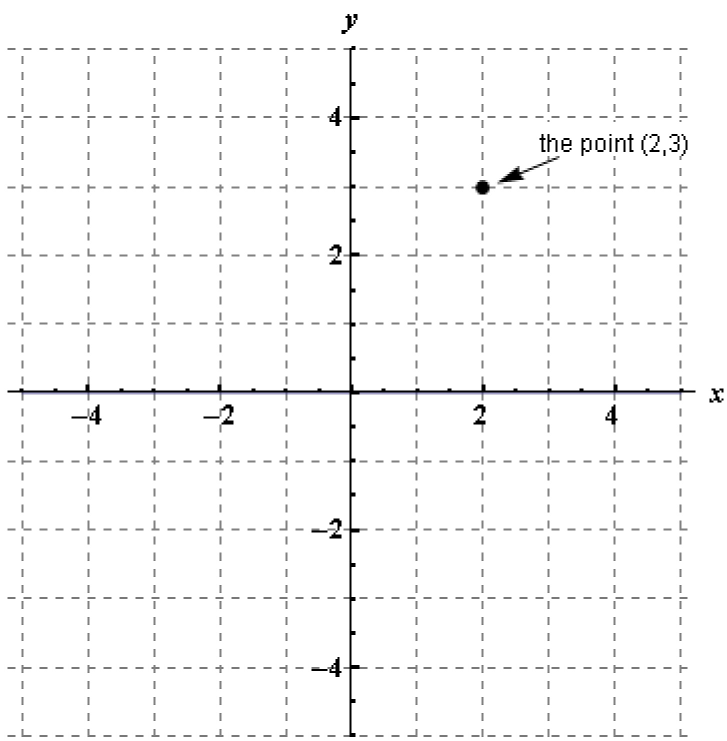
In this supplement we will attempt to answer some of the following questions about straight lines:

I. What do I need to know in order to find such an equation?
II. Are there other ways to express the equation of a straight line?
III. Can I write *any* line in the form $y = mx + b$?
IV. How can I translate a word problem into the equation of a straight line?

These questions will be answered in a style called *proof by example*. It would be helpful for you to try to work out the examples along with the text. You may even want to work out examples using different methods than the ones used here. To review the material, you can use the summary at the end of each section.

I. What do I need to know to find the equation of a straight line?

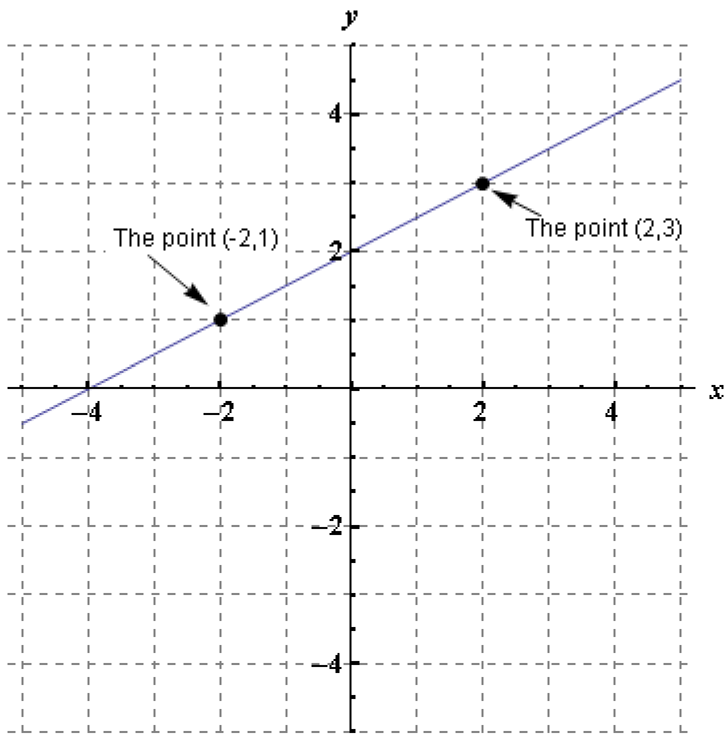
The easiest way to figure this out is to draw a picture and see how many things you need to draw on your picture to figure out a unique line. For example, suppose I were to ask you to find the straight line through the point (2,3):



Now, there are lots of straight lines that I can draw through that point. In fact, there are an *infinite* number of lines that I can draw through that point. Clearly, if I want to find a *unique* straight line, I will need more information.

One thing that is (hopefully) obvious from the picture is that all I would need to determine a unique straight line is another point. This is a famous result from geometry (due to some guy named Euclid) that *two points uniquely define a straight line*.

Let's try this out: Here are the points (2,3) and (-2,1) along with the line that goes through them:



Now we need to find the equation that represents this line. If we wish to speak formally, we can say "given an x-coordinate, what is the corresponding y-coordinate on the line?"

We can start by making the assumption that our line can be written in the form $y = mx + b$. All we need to do is figure out what m is and what b is and we're done!

Let us first make the observation that when *x* is 2, *y* is 3, and also when *x* is -2, *y* is 1. We can write this algebraically as:

$$3 = m(2) + b$$
$$1 = m(-2) + b$$

We can solve for *m* by subtracting the second equation from the first:

$$3 - 1 = (2m + b) - (-2m + b)$$
$$3 - 1 = 2m + b + 2m - b$$
$$2 = 4m$$
$$\frac{2}{4} = m$$
$$\frac{1}{2} = m$$

You could arrive at this same result if you are familiar with the formula for the *slope* of a line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}\text{, where } (x_1, y_1) = (2,3) \text{ and } (x_2, y_2) = (-2,1).$$

Applying this formula gives us: $m = \frac{1-3}{-2-2} = \frac{-2}{-4} = \frac{1}{2}$

Since we now know what *m* is, we can substitute one of our points into the equation and solve for *b*.

$$y = \frac{1}{2}x + b$$
$$3 = \frac{1}{2}(2) + b$$
$$3 = 1 + b$$
$$2 = b$$

The equation of the line is therefore: $y = \frac{1}{2}x + 2$.

Now that I have the equation of this straight line, what do I do with it?

The answer is, of course, find more points on the line!
Some other points I may wish to know are the *intercepts* of the line. The *x-intercept* is the place where the line crosses the *x*-axis (that is, where $y = 0$) and the *y-intercept* is the place where the line crosses the *y*-axis (where $x = 0$.) Let's find these:

***x*-intercept:**

First, let $y = 0$.

$$0 = \frac{1}{2}x + 2$$

Now solve for *x*.

$$-2 = \frac{1}{2}x$$
$$-4 = x$$

The *x*-intercept is therefore the point (-4,0), since when *x* is -4, *y* is zero.

***y*-intercept:**

First, let $x = 0$.

$$y = \frac{1}{2}(0) + 2$$

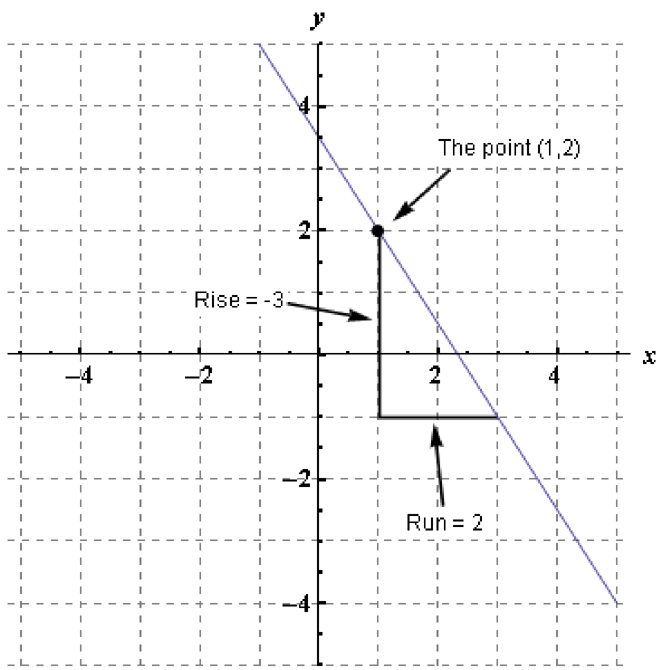
Without any simplification whatsoever, we have $y = 2$. The *y*-intercept is therefore (0,2). You should be able to easily verify this by staring at the graph of the line.

The equation of a line given a point and a slope:
Sometimes rather than finding the equation of a line through two points, we may wish to find the equation of a line through one point that has a particular slope. In this case, we still have enough information to find the equation of the line.

As an example, let's find the equation of the line that goes through the point (1,2) and has a slope of -3/2.

For a graph of this line, we can start at the point (1,2) and use the slope to help us find a new point. You may recall that the slope is referred to as the *rise* over the *run* of the graph. So if the slope is expressed as a fraction, just make the *rise* the numerator, and the *run* the denominator.

Rise = -3 (This means move down 3 units from the starting point.)
Run = 2 (This means move right 2 units)



Finding the equation of this line should be easy, since we already know what the slope is (m = -3/2.)

$$y = -\frac{3}{2}x + b$$

$$2 = -\frac{3}{2}(1) + b$$

$$\frac{7}{2} = b$$

The equation of the line is therefore: $y = -\frac{3}{2}x + \frac{7}{2}$

Summary

1. In order to find the equation of a straight line you need to know

a) Two points on the line, or
b) The slope of the line and one point on the line.

2. The equation of a straight line is written in the *slope-intercept form* as:

$$y = mx + b$$

Where m is the slope of the line and b is the y-intercept.

3. The slope of a line that goes through the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope can be graphically interpreted as:

$$slope = \frac{rise}{run}$$

4. Once the slope of the line is known, the y-intercept b can be found by substituting one of the points into the equation of the line.

5. Once the equation of the line is known, any other point on the line can be found by either specifying a value for x and finding y or specifying a value for y and finding x.

II. Some other ways to express the equation of a straight line

We saw in the first section the equation of a line written in *slope-intercept form*. Here we will take a look at some other ways to write the equation of a straight line.

Point-slope form:

A line with slope m that goes through a point (x_1, y_1) can be written as:

$$y - y_1 = m(x - x_1)$$

This form is convenient to use if you want to write down the equation of a line very quickly. For example:

To find the equation of the line with a slope of 2 that goes through the point (1,3), we can directly plug this information into the point slope form to get:

$$y - 3 = 2(x - 1)$$

That's it! If you prefer to write this in slope-intercept form we can just solve for y:

$$y - 3 = 2(x - 1)$$
$$y - 3 = 2x - 2$$
$$y = 2x + 1$$

We now have an alternative way to find the equation of a straight line which is sometimes much faster to compute.

An added benefit of the point-slope form becomes apparent if we ask a different sort of question about the behavior of the line:
How much does y change if I change x by a certain amount?

If we call the "change in x" $\Delta x = x - x_1$ and the "change in y" $\Delta y = y - y_1$ we can rewrite the equation of the line as:

$$\Delta y = m\Delta x$$

To put this in simpler terms, we can say that *the change in y is equal to the slope times the change in x*. If you think about it, this is really just another way of saying that *the slope is equal to the change in y divided by the change in x* or *the slope is the rise over the run!*

In working out problems involving a linear relationship between two quantities, this is often the more intuitive way to think. If x increases be some amount Δx, then y will increase by $m\Delta x$.

The general form of the equation of a straight line
Yet another way to write the equation of a straight line is the form:

$$Ax + By = C$$

This, as the name might suggest, is the most general way to express the equation of a straight line (we'll get into why that is in the next section.) The important thing is that if you see an equation like this, you should recognize that it is a straight line.

As with the point-slope form, this form can be solved for *y* and put into the slope-intercept form.

For example, consider the line:

$$2x - 3y = 5$$

If we solve this equation for *y*, we get:

$$y = \frac{2}{3}x - \frac{5}{3}$$

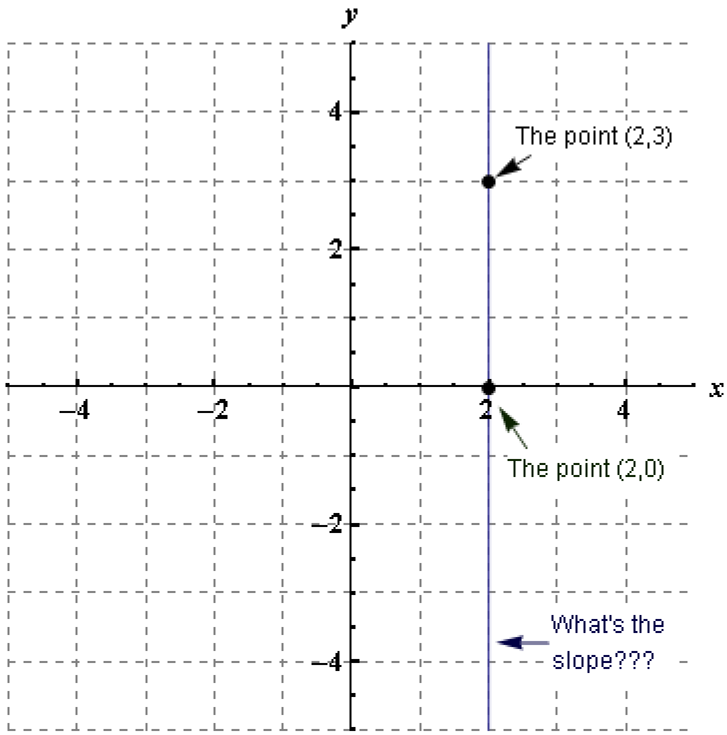
Summary
The different forms for the equation of a straight line are in some sense arbitrary and subject to personal preference. It is important to recognize the equivalence between these forms, and how to convert an equation from one form into another. The lines you will be given in practical problems will not always be written in your favorite form at first, but with the knowledge that each is just a different way of saying the same thing, you can convert it into your favorite form!

The different forms:

1. Slope-intercept form: $y = mx + b$
2. Point-slope form: $y - y_1 = m(x - x_1)$
3. General form: $Ax + By = C$

III. Can I write the equation of *any* line in the form $y = mx + b$?

To answer this question we will look at the following situation: What is the slope of the line that goes through the points (2,0) and (2,3)?



If we use the regular slope formula to find the slope of this line, we immediately run into trouble:

$$m = \frac{3-0}{2-2}$$
$$m = \frac{3}{0}$$

Dividing by zero is not an allowed operation! If we were to allow it, we could prove all kinds of crazy statements like 1 = 2. We must conclude that the slope of this line is *undefined*.

On the other hand, this is clearly a perfectly defined line. It is easy to draw, so there must be some way of expressing the equation of this line algebraically. If we note that every point on the line has the same x-coordinate (namely, 2) then we can come up with our equation:

$$x = 2$$

So even though the slope is *undefined*, we can still come up with an equation for the line, just not in any form that must make use of the slope. As it turns out, the general form for the equation of a straight line is better suited to handle this situation.

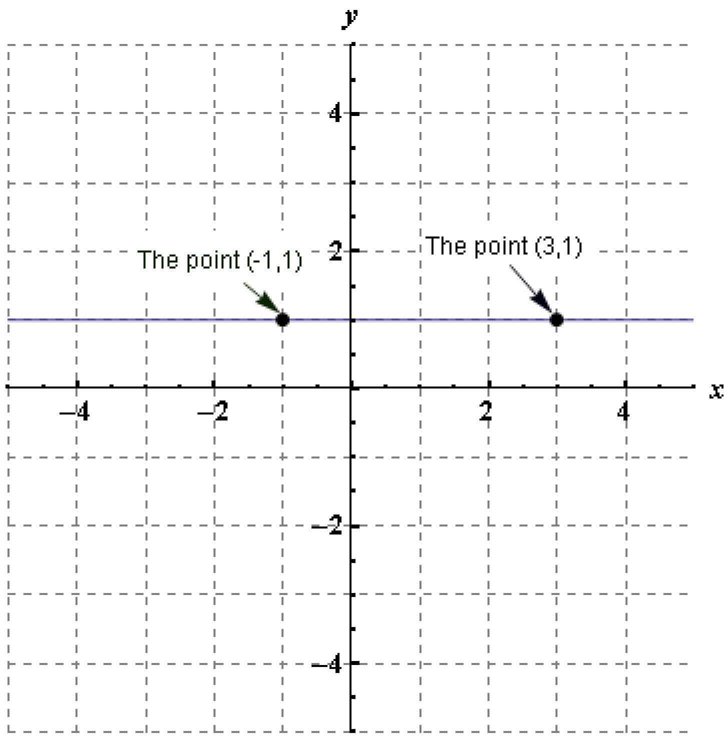
$$Ax + By = C$$

If we let $A = 1$, $B = 0$, and $C = 2$, we have the equation of our vertical line. The conclusion here is that the slope-intercept form works to describe every type of line *except* vertical ones. Vertical lines can be described by the formula:

$x = k$, where k is some constant.

What about horizontal lines?

Let's take a look at an example: Consider the line that goes through the points (-1,1) and (3,1):



We can use the slope formula to find the slope of this line:

$$m = \frac{1-1}{3-(-1)}$$

$$m = \frac{0}{4}$$

$$m = 0$$

Zero is an acceptable slope, so everything here is fine. The equation of the line is then:

$y = 0x + b$, or just $y = b$.

Since the y-coordinate is 1 no matter what the x-coordinate is, we can say that $b = 1$, or the equation of the line is:

$y = 1$

The conclusion is that a horizontal line has a slope of zero, and can be expressed in the form:

$y = k$, where k is some constant.

Summary:

Vertical lines have an undefined slope, but can be expressed by the equation:

$x = k$

Horizontal lines have a slope of zero, and can be expressed by the equation:

$y = k$

The slope-intercept ($y = mx + b$) and point-slope ($y - y_1 = m(x - x_1)$) form for the equation of a straight line can be used to describe any lines that are not vertical.

The general form for the equation of a straight line ($Ax + By = C$) can be used to describe *any* line, including vertical ones.

IV. Translating word problems into linear equations

Units:
In order to make sense of a word problem, we need a way to mathematically describe what each quantity represents, and to relate quantities of different types.

Here the notion of *the units of a quantity* is very powerful. First we need some rules governing these things:

1) Two quantities can be added together if and only if they have the same units.
2) Two quantities can be set equal to each other if and only if they have the same units.
3) If I multiply two quantities, I multiply their units
4) If I divide two quantities, I divide their units

Rules 1 and 2 can be considered the apples and oranges rules: You can add apples to apples, or oranges to oranges, but not apples to oranges. Also, if I have 5 apples, I cannot say that I have 5 oranges.

Rules 3 and 4 behave nicely together by making the multiplication of units very similar to the multiplication of fractions. We'll take a look at some simple examples here, but this is a very deep topic itself, and carries around with it a fancy title called *dimensional analysis*.

Distance equals velocity times time:
This is an equation you have probably implicitly used many times in your life, with absolutely no reference to a math class.

Consider the following situation:

> Yesterday, I drove my car at a speed of 50 miles per hour for two hours. How far did I drive?

Hopefully, your immediate response to this question is "100 miles!" Let's examine how we arrive at this conclusion:

First, in the world of units, the word "per" denotes division. So the units of "miles per hour" should be written as $\frac{miles}{hour}$.

Second, we can see that we arrived at the answer by multiplying the velocity and the time:

$$50\frac{miles}{hour}\times 2hours = 100miles$$

The *hours* units canceled out just as if we were multiplying fractions!

What does this have to do with the equation of a straight line?
To include the notion of a *linear equation*, we should consider a more general situation:

> Yesterday, I drove my car at a speed of 50 miles per hour for *x* hours. How far did I drive?

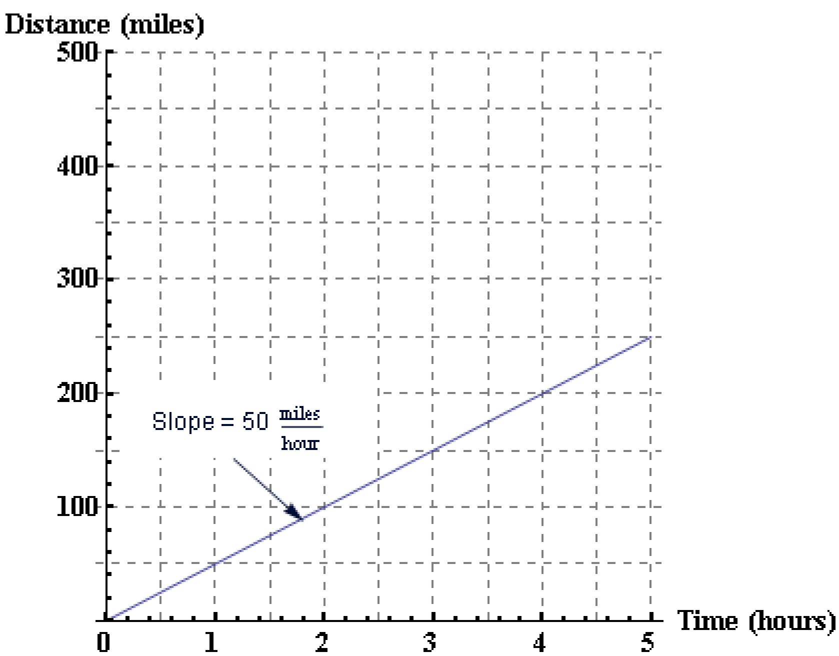
There was nothing really special about the number 2 in the first scenario, so all we need to do is replace the 2 with *x* and we can find out how far I drove after any number of hours.

$$50\frac{miles}{hour} \times x \cdot hours = 50x \cdot miles$$

If we let *y* be the distance (in miles) I drove in *x* hours, we then get the equation:

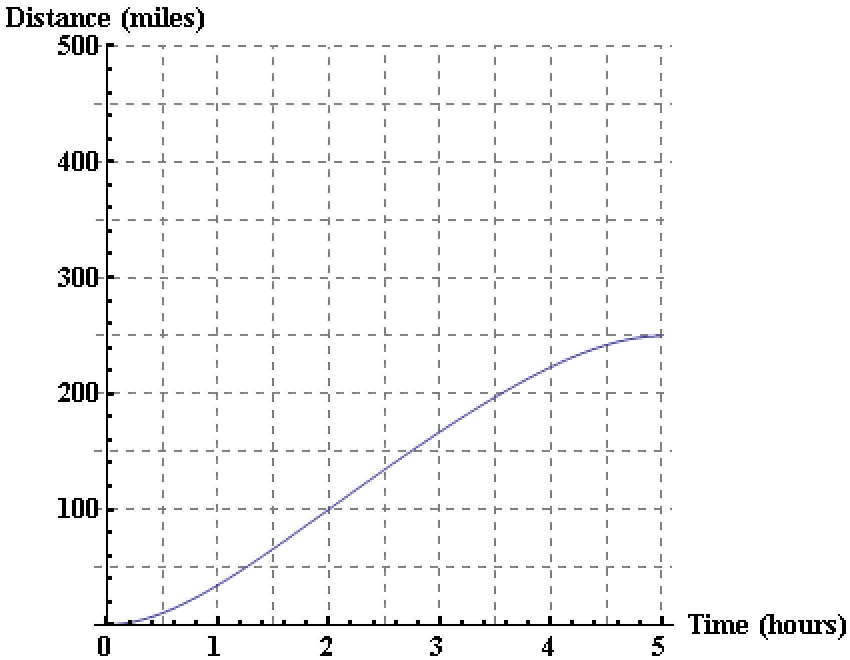
$$y \cdot miles = 50x \cdot miles \text{ or } y = 50x.$$

This is the equation of a line! Let's see a graph of this:



We should note that the *slope* of this graph is simply the velocity, and the units of the slope are the units of *y* (miles) divided by the units of *x* (hours.) If we also note that velocity can be interpreted as the *rate* distance changes with respect to time, we can conclude that, in general, the slope of a line can be interpreted as the *rate of change* of *y* with respect to *x*.

What if I wasn't driving at a constant velocity?
Anyone with a little experience driving ought to know that the speed of the car is not the same at any given time. If you watch your speedometer as you drive you would notice that it steadily increases as you speed up and steadily decreases as you slow down. Certainly the graph representing distance vs. time should be more complicated than the straight line we saw in the previous example! It should probably look more like this:



How do we think about the velocity of the car here? Let's go back to the original statement of "Yesterday, I drove my car at a speed of 50 miles per hour for two hours," and try to express this a little more clearly to reflect a more realistic situation.

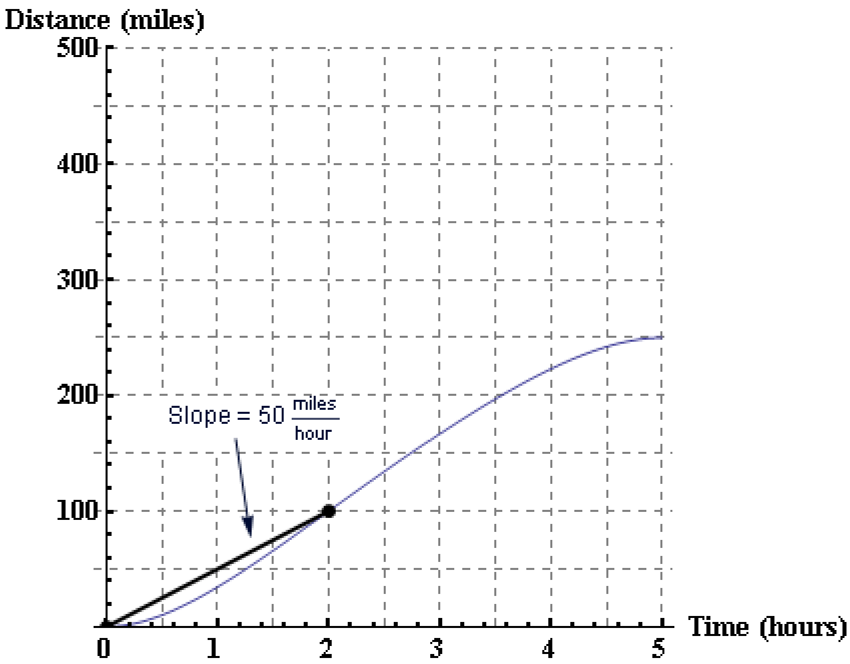
Hidden in this statement is the assumption that my speedometer read 50 miles per hour the entire time I was driving. What I should have said is that my *average* velocity was 50 miles per hour, I may have been going faster some of the time, and slower some of the time.

If we flip this question on its head and ask "If I drove 100 miles in two hours yesterday, what was my *average* velocity?" we can see that this question doesn't depend on the specifics of when I was going faster or slower than 50 miles per hour.

With this in mind, we can define the *average velocity* as $\frac{dis\tan ce \cdot traveled}{elapsed \cdot time}$.

This can be interpreted graphically as the slope of the line between two points on the distance vs. time graph.

For example:
On the previous graph we would like to know the average velocity over the first two hours of driving (that is, between $x = 0$ and $x = 2$.) To do so, we just draw a straight line between those two points on the graph and find the slope.



Even though I wasn't driving 50 miles per hour the entire time, we can see that my average velocity was 50 miles per hour, and for many purposes that is good enough to describe my trip.

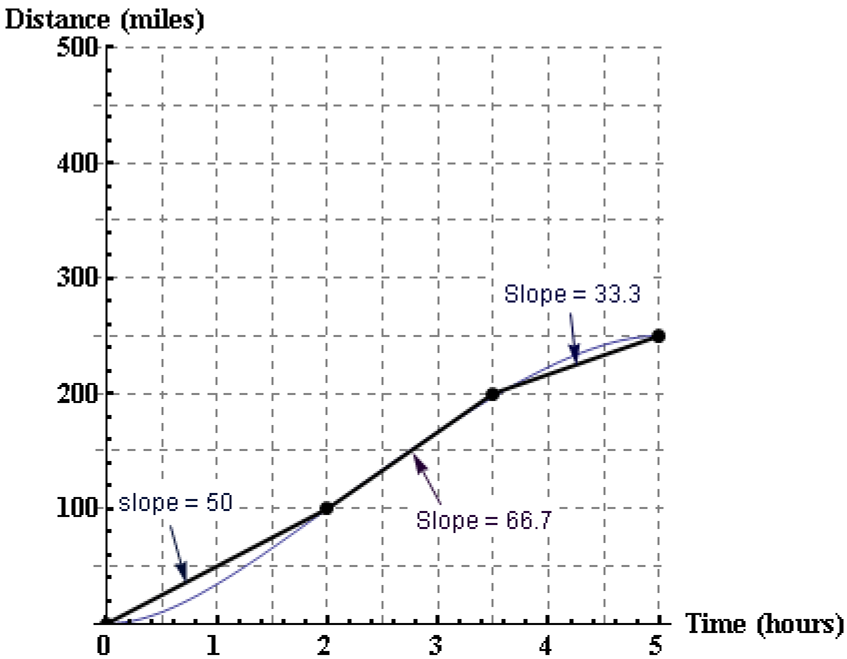
Let's make this a bit more formal:

a) A *secant line* is a line that connects two points on a curve.
b) The *slope* of a secant line is the average slope of the curve between those two points.
c) The slope of a secant line on a distance vs. time graph is the *average velocity* during the time interval represented.

The formula for the slope of a secant line is the same as the formula for the slope of any other line, namely:

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Finding average velocity on different intervals:
Looking at the graph a little more closely, we can see that the car is moving a little faster after 2 hours, and starts slowing down after about 3.5 hours. We can use the same procedure to find the average velocity between $x = 2$ and $x = 3.5$, and also from $x = 3.5$ to $x = 5$:



Splitting up a graph into pieces like this is useful if the velocity changes significantly from one time interval to the next. It is a judgment call as to where each interval begins and ends. The choice depends on what the actual graph looks like, and how many different lines you wish to draw. For the best accuracy, it is better to have more lines with shorter intervals, but this comes at the expense of having to perform more calculations.

It must also be stressed that we are using the secant lines to *approximate* the curve, and that the information we gain from these approximations are only estimates. Unfortunately, when making the leap from purely mathematical problems to messy, real life problems, such approximations are necessary.

In other words, math is usually neat and tidy; it’s real life that causes a mess.

Creating a straight line from a set of data:
Suppose you are out at sea, and you wish to know what the pressure under the water is at a particular depth. The reasons why you would want to know are unimportant for this discussion (although scuba divers who want to know how deep they can go before their eardrums collapse might disagree!) The way we could figure this out is to measure the pressure at various depths, and derive an equation from all of the measurements.

Hopefully, after all of the measurements are taken and plotted on a graph, the data points will look like they form a straight line. If they do, then we can make predictions about the pressure at depths greater than the ones we measured.

On the next page we have a table of such measurements. The pressure (in pascals, one of the many units used to measure pressure) is recorded at various depths (in meters.)

Some preliminaries:
You probably should not get too tied down to always using x and y as your variables. They are just letters. We can use other letters that better describe the problem at hand.

Here, it seems reasonable to use d for depth and P for pressure.

We want to find out if we can represent the data in the table by a linear equation:

$$P = md + b$$

The formula for the slope here would be:

$$m = \frac{P_2 - P_1}{d_2 - d_1} = \frac{\Delta P}{\Delta d}$$

An important property of straight lines is that the slope is the same *no matter which two points you use to calculate it.*

So if the slope between any two data points in our table is the same (or very nearly the same) we can model the data as a linear equation.

Diver Down

Pressure/Depth Chart
(up to 500 meters)

Depth (meters)	Pressure (pascals)	Pressure (Kpa)
0	100,000	100
5	149,000	149
10	198,000	198
15	247,000	247
20	296,000	296
25	345,000	345
30	394,000	394
35	443,000	443
40	492,000	492
45	541,000	541
50	590,000	590
55	639,000	639
60	688,000	688
65	737,000	737
70	786,000	786
75	835,000	835
80	884,000	884
85	933,000	933
90	982,000	982
95	1,031,000	1,031
100	1,080,000	1,080
200	2,060,000	2,060
500	5,000,000	5,000

Calculating the slopes:
It would require too much space here to compute the slope between *every* pair of data points in the table, so we'll just use a few:

Slope between 0 meters and 5 meters:

$$m = \frac{149000 - 100000}{5 - 0} = 9800$$

Slope between 5 meters and 10 meters:

$$m = \frac{198000 - 149000}{10 - 5} = 9800$$

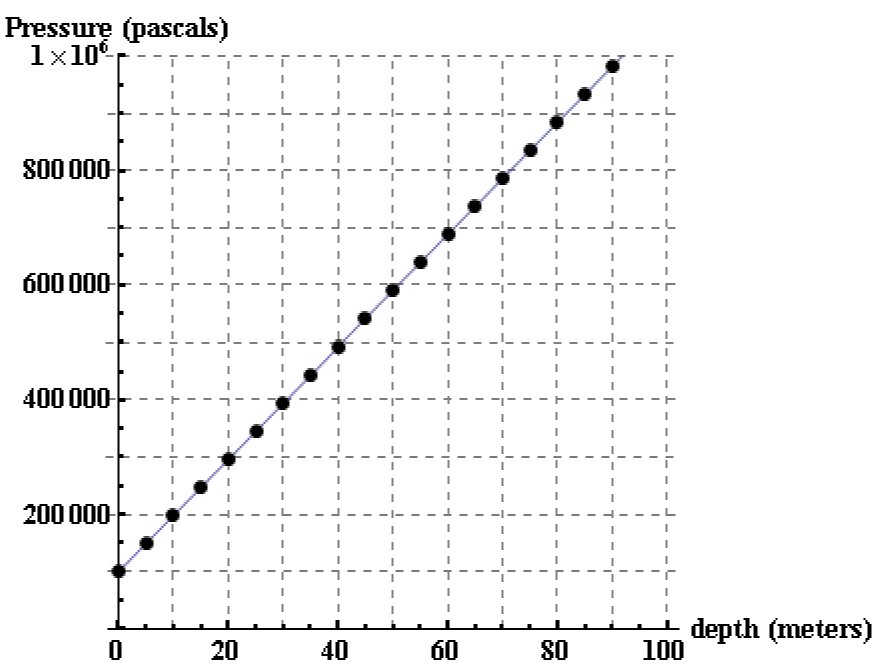
Slope between 10 meters and 50 meters:

$$m = \frac{590000 - 198000}{50 - 10} = 9800$$

Slope between 50 meters and 500 meters:

$$m = \frac{5000000 - 590000}{500 - 50} = 9800$$

If doing these calculations numerically is not convincing, plotting all of the points on a graph should be:



What's the equation of the line?
We have already calculated the slope of this line. In fact, we had to calculate the slope many times in order to convince ourselves that it was a line in the first place.

$$slope = 9800\frac{pascals}{meter}$$

Note that the units of the slope are simply the units of pressure (on the *y*-axis) divided by the units of distance (on the *x*-axis.) The physical interpretation of this is simply that the pressure increases at a rate of 9800 pascals per meter.

Since we know the slope, we can just substitute a point into our general equation for the line. The point (0, 100000) fits the bill nicely.

$$P = 9800d + b$$
$$100000 = 9800(0) + b$$
$$100000 = b$$

To make all of the units consistent, the units of *b* should be *pascals*.
The equation of our line is now:

$$P = 9800d + 100000$$

We can now figure out the pressure at *any* depth.

Summary:

Rules for units:

1) Two quantities can be added together if and only if they have the same units.
2) Two quantities can be set equal to each other if and only if they have the same units.
3) If I multiply two quantities, I multiply their units
4) If I divide two quantities, I divide their units

The units of the slope of a straight line are the units of *y* divided by the units of *x*.

The *average rate of change* of a graph between two points is the slope of the secant line connecting those two points.

$$\textit{Average rate of change} = \frac{y_2 - y_1}{x_2 - x_1}$$

The *average velocity* of an object between times t_1 and t_2 is:

$$v_{ave} = \frac{d_2 - d_1}{t_2 - t_1}$$, where d_1 and d_2 are the positions at t_1 and t_2, respectively.

More intuitively, the average velocity is the distance traveled divided by the elapsed time.

A set of data can be modeled by a linear equation if the slope of the lines connecting any two data points are all (approximately) the same.